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Effectiveness of the Daftardar-Jafari Method in Solving Higher-Order Ordinary Differential Equations: A Numerical Study

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Abstract: *This study explores the Daftardar-Jafari Method (DJM) for solving linear and nonlinear higher-order ordinary differential equations (ODEs). Unlike traditional perturbation methods, DJM does not rely on small parameters, making it highly effective for strongly nonlinear problems. The method constructs a rapidly converging iterative sequence, yielding accurate analytical or approximate solutions with reduced computational costs. We applied DJM to a range of benchmark problems and compared the results with those obtained using the Homotopy Perturbation Method (HPM). The DJM provided significantly higher accuracy, demonstrating its superior performance in terms of convergence and computational efficiency. The numerical results, computed using Maple software, reinforce the practical advantages of DJM for solving complex higher-order ordinary differential equations. In conclusion, DJM is an effective and efficient tool for solving a broad class of higher-order ordinary differential equations, outperforming traditional methods in terms of accuracy and reliability*.

Keywords: Linear and nonlinear equation, higher-order ordinary differential equation, Daftardar-Jafari Method, Exact and approximate solution.

I. INTRODUCTION

Ordinary differential equations (ODEs), both linear and nonlinear, including higher-order forms [1], play a fundamental role in modeling and analyzing dynamic systems across various scientific and engineering disciplines. These equations describe a wide range of phenomena, including mechanical vibrations, fluid flow, heat transfer, wave propagation, and control processes in robotics and automation. Higher-order ODEs frequently arise in structural mechanics, aerospace engineering, and electromechanical systems, where accurate solutions are essential for optimizing designs and ensuring system stability. The complexity of solving linear and nonlinear higher-order ODEs has long been a challenge, as traditional analytical methods often fail to provide explicit solutions. Consequently, researchers have focused on developing efficient semi-analytical and iterative techniques that combine analytical accuracy with numerical adaptability. Among these, the Variational Iteration Method (VIM) [2,3], the Adomian Decomposition Method (ADM) [4-6], and the Homotopy Perturbation Method (HPM) [7-14] have gained significant attention due to their ability to approximate solutions without requiring linearization or small perturbation parameters. The Variational Iteration Method (VIM) constructs iterative correction functionals to refine approximations progressively, making it effective for a wide class of differential equations. However, its convergence speed depends on the choice of initial approximations. The Adomian Decomposition Method (ADM) decomposes nonlinear terms into infinite series of polynomials, simplifying the problem into a sequence of solvable components. Despite its effectiveness, ADM requires complex polynomial calculations, which may limit its efficiency for highly nonlinear problems. The Homotopy Perturbation Method (HPM) combines homotopy theory with perturbation techniques, avoiding small parameter dependencies. While HPM is widely used, it may still struggle with strongly nonlinear systems when the convergence rate is slow. Amidst these techniques, the Daftardar-Jafari Method (DJM) [15-17] has emerged as a powerful alternative for solving linear and nonlinear higher-order ODEs. Unlike traditional perturbation-based approaches, DJM does not rely on small parameters or complex transformations, making it highly efficient for nonlinear systems. The method constructs a rapidly converging iterative sequence, ensuring high accuracy while reducing computational complexity. DJM has been successfully applied to various benchmark problems, demonstrating superior performance in comparison to other semi-analytical methods. Its ability to solve nonlinear ODEs with minimal computational effort makes it a valuable tool for researchers and engineers alike. This study aims to explore the effectiveness of DJM in solving linear and nonlinear higher-order ODEs, emphasizing its precision and computational advantages. Through comparative analysis, we highlight how DJM outperforms other methods in terms of convergence rate, accuracy, and efficiency. The findings of this research reinforce DJM's applicability in scientific and engineering fields, making it a promising approach for tackling complex nonlinear differential equations. This paper is organized as follows: Section 2 provides an overview of the mathematical framework and theoretical principles underlying the Daftardar-Jafari Method. Section 3 presents the application of DJM to various types of linear and nonlinear higher-order ordinary differential equations, including numerical results and comparative analysis. Finally, Section 4 summarizes the main findings of the study and discusses potential future research directions, highlighting areas where DJM can be further improved and extended to more complex problems.

II. METHODOLOGY AND FORMULATION

We focus on the following initial value problem that expressed in the form:

$$u^{(n)}(x) = f\left(x, u, u', \dots, u^{(n-1)}\right) + g(x)\ \ ,, x \in [0, a] \quad (1)$$

Where u is unknown function and g is analytical function.

With the initial conditions

$$u(0) = \alpha_0\ , u'(0) = \alpha_1, \dots, u^{(n-1)}(0) = \alpha_{n-1} \quad , \quad \alpha_i \in R\ , i = \overline{0, n-1}$$

According to the DJM, by integrating both sides of the system (1) with respect to x

$$u(x) = \underbrace{\int_0^x \dots \int_0^x}_{n} \left(f\left(x, u, u', \dots, u^{(n-1)}\right) + g(x)\right) \underbrace{dx \dots dx}_{n}$$

(2)

Where the nonlinear part

$$N(x) = \underbrace{\int_0^x \dots \int_0^x}_{n} \left(f\left(x, u, u', \dots, u^{(n-1)}\right) + g(x)\right) \underbrace{dx \dots dx}_{n}$$

And the initial solution is

$$u_0(x) = \sum_{i=0}^{n-1} \alpha_i \frac{x^i}{i!} + \underbrace{\int_0^x \dots \int_0^x}_{n} \left(g(x)\right) \underbrace{dx \dots dx}_{n}$$

The solution to the (2) is given in the form

$$u = \sum_{i=0}^{\infty} u_i \quad (3)$$

By substituting (3) into (2), yields

$$\sum_{i=0}^{\infty} u_i(x) = \underbrace{\int_0^x \dots \int_0^x}_{n} \left(f\left(x, \sum_{i=0}^{\infty} u_i, \sum_{i=0}^{\infty} u'_i, \dots, \sum_{i=0}^{\infty} u_i^{(n-1)}\right) + g(x)\right) \underbrace{dx \dots dx}_{n}$$

From this, the following results:

$$u_0(x) = \sum_{i=0}^{n-1} \alpha_i \frac{x^i}{i!} + \underbrace{\int_0^x \dots \int_0^x}_{n} \left(g(x)\right) \underbrace{dx \dots dx}_{n}$$

$$u_1(x) = N(u_0)$$

$$u_2(x) = N(u_0 + u_1) - N(u_0)$$

$$u_m(x) = N(u_0 + \dots + u_{m-1}) - N(u_0 + \dots + u_{m-2}), \quad m = 2, 3, \dots$$

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For convergence of the DJM [17]

Lemma [18]. If N is C^{∞} in a neighborhood of u_0 and $\left\|N^{(n)}(u_0)\right\| < L$, for any n and for some real $L > 0$ and $\|u_i\| \le M < e^{-1}, i = 1,2, \dots$ then the series $\sum_{n=0}^{\infty} u_n$ is absolutely convergent and

$$\|u_i\| \le L M^n e^{n-1}(e-1), \ \ n = 1,2, \dots$$

III. TEST PROBLEMS

Example 1

Consider the following ninth-order initial value problem [1]

$$u^{(9)}(x) = -9e^x + u(x)\,, \quad 0 \le x \le 1 \quad (4)$$

Subjects to the initial conditions:

$$u^{(i)}(0) = 1 - i\,, \quad i = \overline{0,8}$$

According to the DJM, by integrating both sides of the equation (4) with respect to x and by using the boundary conditions, yields

$$u(x) = 10 + 9x + 4x^2 + \frac{7x^3}{6} + \frac{x^4}{4} + \frac{x^5}{24} + \frac{x^6}{180} + \frac{x^7}{1680} + \frac{x^8}{20160} - 9e^x + \int_0^x \dots \int_0^x u(x)\, dx \dots dx$$

Where

$$u_0(x) = 10 + 9x + 4x^2 + \frac{7x^3}{6} + \frac{x^4}{4} + \frac{x^5}{24} + \frac{x^6}{180} + \frac{x^7}{1680} + \frac{x^8}{20160} - 9e^x$$

$$\& N(u) = \int_0^x \dots \int_0^x u(x)\, dx \dots dx$$

Then

$$u_1(x) = N(u_0) = \int_0^x \dots \int_0^x u_0(x)\, dx \dots dx = 9 + 9x + \frac{9x^2}{2} + \frac{3x^3}{2} + \frac{3x^4}{8} + \frac{3x^5}{40} + \frac{x^6}{80} + \frac{x^7}{560} + \cdots$$

$$u_2(x) = N(u_0 + u_1) - N(u_0) = 9 + 9x + \cdots + \frac{x^8}{4480} + \frac{x^9}{40320} + \cdots - 9e^x$$

Then the exact solution is

$$u(x) = \sum_{i=0}^{\infty} u_i$$

Table 1. shows the 'one term' , 'two terms' and 'three terms' refer to the number of terms used in the solution using the Daftardar-Jafari Method, while the Homotopy perturbation Method required 'twelve terms' in the series expansion for $u(x)$

Where the exact solution of (4) is $u(x) = (1-x)e^x$

x	*Error of DJM* u_0 $n=1$	*Error of DJM* $u_0 + u_1$ $n=2$	*Error of DJM* $u_0 + u_1 + u_2$ $n=3$	*Error of HPM* [1] $u_0 + \cdots + u_{11}$ $n=12$
0	0	0	0	0
0.1	5.1×10^{-15}	2.2×10^{-18}	1.3×10^{-20}	3.6×10^{-9}
0.2	4.2×10^{-12}	7.3×10^{-17}	1.0×10^{-19}	3.4×10^{-9}
0.3	1.1×10^{-11}	4.8×10^{-17}	3.8×10^{-17}	4.6×10^{-9}
0.4	3.5×10^{-10}	5.2×10^{-16}	2.3×10^{-16}	1.4×10^{-9}
0.5	1.5×10^{-9}	9.5×10^{-15}	1.4×10^{-16}	4.5×10^{-9}

Example 2

Consider the following nonlinear tenth-order initial value problem [1]

$$u^{(10)}(x) = e^{-x}u^2(x)\,, \quad 0 \le x \le 0.5 \qquad (5)$$

Subjects to the initial conditions:

$$u^{(i)}(0) = 1\,, \quad i = \overline{0,9}$$

According to the DJM, by integrating both sides of the equation (5) with respect to x and by using the boundary conditions, yields

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \int_0^x \dots \int_0^x e^{-x}u^2(x)\,dx \dots dx$$

Where

$$u_0(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880}$$

$$\& N(u) = \int_0^x \dots \int_0^x e^{-x}u^2(x)\,dx \dots dx$$

Then

$$u_1(x) = N(u_0) = \int_0^x \dots \int_0^x e^{-x}u_0^2(x)\,dx \dots dx = 172577535989x - \frac{55295895511x^2}{2} + \frac{5397936163x^3}{2} + \cdots$$

$$u_2(x) = N(u_0 + u_1) - N(u_0) = \frac{15480158855284131389 5x^4e^{-x}}{12} + \cdots$$

Then the exact solution is

$$u(x) = \sum_{i=0}^{\infty} u_i$$

Table 1. shows the 'one term' , 'two terms' and 'three terms' refer to the number of terms used in the solution using the Daftardar-Jafari Method, while the Homotopy perturbation Method required 'twelve terms' in the series expansion for $u(x)$

Where the exact solution of (5) is $u(x) = e^x$

x	Error of DJM u_0 $n = 1$	Error of DJM $u_0 + u_1$ $n = 2$	Error of DJM $u_0 + u_1 + u_2$ $n = 3$	Error of HPM [1] $u_0 + \cdots + u_{11}$ $n = 12$
0	0	0	0	0
0.1	3.2×10^{-16}	1.2×10^{-17}	3.4×10^{-19}	1.4×10^{-6}
0.2	4.1×10^{-14}	6.8×10^{-17}	4.5×10^{-18}	2.7×10^{-6}
0.3	2.8×10^{-12}	8.1×10^{-16}	9.2×10^{-18}	3.7×10^{-6}
0.4	3.9×10^{-11}	1.6×10^{-16}	7.2×10^{-17}	4.4×10^{-6}
0.5	8.7×10^{-10}	1.0×10^{-15}	1.5×10^{-17}	4.5×10^{-6}

Example 3

Consider the following nonlinear tenth-order initial value problem [1]

$$u^{(12)}(x) = 2e^{x}u^2(x) + u^{(3)}(x)\,, \quad 0 \le x \le 0.5 \qquad (6)$$

Subjects to the initial conditions:

$$u^{(i)}(0) = (-1)^i\,, \quad i = \overline{0,11}$$

According to the DJM, by integrating both sides of the equation (6) with respect to x and by using the boundary conditions, yields

$$u(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \frac{x^8}{40320} - \frac{x^9}{362880} + \frac{x^{10}}{3628800} - \frac{x^{11}}{39916800} + \int_0^x \dots \int_0^x 2e^{x}u^2(x) + u^{(3)}(x)\,dx \dots dx$$

Where

$$u_0(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \frac{x^8}{40320} - \frac{x^9}{362880} + \frac{x^{10}}{3628800} - \frac{x^{11}}{39916800}$$

$$\& N(u) = \int_0^x \dots \int_0^x 2e^{x}u^2(x) + u^{(3)}(x)\,dx \dots dx$$

Then

$$u_1(x) = N(u_0) = \int_0^x \dots \int_0^x 2e^{x}u_0^2(x) + u_0^{(3)}(x)\,dx \dots dx = 2064138802538x - 33320055241771x^2 + \cdots$$

Then the exact solution is

$$u(x) = \sum_{i=0}^{\infty} u_i$$

Table 1. shows the 'one term' , 'two terms' and 'three terms' refer to the number of terms used in the solution using the Daftardar-Jafari Method, while the Homotopy perturbation Method required 'twelve terms' in the series expansion for $u(x)$

Where the exact solution of (6) is $u(x) = e^{-x}$

x	Error of DJM u_0 $n = 1$	Error of DJM $u_0 + u_1$ $n = 2$	Error of HPM [1] $u_0 + \cdots + u_{11}$ $n = 12$
0	0	0	0
0.1	3.6×10^{-16}	3.4×10^{-18}	3.5×10^{-7}
0.2	1.2×10^{-16}	6.5×10^{-18}	6.1×10^{-7}
0.3	9.2×10^{-15}	1.2×10^{-17}	9.9×10^{-7}
0.4	9.6×10^{-14}	2.1×10^{-17}	7.2×10^{-7}
0.5	8.2×10^{-13}	4.9×10^{-16}	1.5×10^{-7}

One of the key advantages of the Daftardar-Jafari Method (DJM) is its exceptional convergence rate, which dramatically reduces the number of iterations required to obtain an accurate solution compared to traditional techniques, such as the Homotopy Perturbation Method (HPM). In our numerical tests, DJM consistently demonstrated faster convergence and higher accuracy, particularly for complex nonlinear systems. Moreover, DJM proved to be more computationally efficient, requiring less time to reach the desired precision compared to HPM, which often faced challenges with higher-order nonlinearities. This outstanding performance in both convergence speed and computational cost positions DJM as an ideal method for solving nonlinear ordinary differential equations, especially in practical applications where computational resources are limited.

IV. CONCLUSION AND DISCUSSION

In this study, the Daftardar-Jafari Method (DJM) was employed to solve higher-order linear and nonlinear ordinary differential equations (ODEs). The findings confirm that DJM delivers highly accurate solutions while significantly reducing computational complexity compared to conventional approaches. By generating a rapidly converging iterative sequence, DJM ensures high precision with fewer iterations, making it a powerful and efficient method for tackling complex mathematical problems. The effectiveness of DJM was validated through various benchmark problems, where it demonstrated superior performance in solving intricate nonlinear ODEs. Additionally, a comparative analysis with the Homotopy Perturbation Method (HPM) highlighted DJM's enhanced accuracy and efficiency. Given its simplicity, fast convergence, and broad applicability, DJM emerges as a reliable tool for solving higher-order differential equations in scientific and engineering domains. Future research could extend DJM's applications to even more complex differential equations, exploring its theoretical foundations and convergence properties in greater depth to further optimize its performance and applicability.

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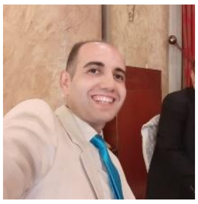

Dr. George Albert Toma holds a Ph.D. in Applied Mathematics and Computer Science from the Higher Institute for Applied Sciences and Technology in the Syrian Arab Republic. He is currently a Professor at the Institute, with more than 15 years of academic experience. He has published approximately 33 research papers in reputable international journals and conferences. He has also reviewed over 150 scientific works, including journal articles, books, and Ph.D. dissertations.

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