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Parameter-uniform hybrid numerical scheme for singularly perturbed initial value problem

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Abstract: *This paper deals with a singularly perturbed initial value problem which depends on a parameter. A hybrid scheme has been constructed by combining a second order cubic spline on the layer region and a midpoint upwind scheme on the smooth region. It is shown that the order of convergence of the proposed method is two in the discrete norm. Error bounds for the numerical solution and its numerical derivative are established. A numerical example is presented which support the theoretical results.*

Keywords: Singular perturbation, Hybrid scheme, Shishkin mesh, Initial value problem.

I. INTRODUCTION

Consider the initial value problem (IVP):

$$\begin{cases} Ly(x) = \varepsilon^2 y''(x) + \varepsilon p(x) y'(x) + q(x) y(x) = g(x), \quad x \in \Omega = (0, X] \\ \qquad y(0) = \lambda, \; L_0 y(0) = y'(0) = \dfrac{\eta}{\varepsilon}, \end{cases} \tag{1}$$

Where, $0 < \varepsilon << 1$, is a perturbation parameter, η and λ are given constant. $p(x)$, $q(x)$ and $g(x)$ are smooth function in $(0, X]$, with $0 < \alpha \le p(x), \; 0 < \beta \le q(x) \le \beta^*$.

Taking the above assumptions into consideration, $y(x)$ has an exponential layer near $x = 0$. This type of model problem found in quantum mechanics, fluid dynamics and other applied areas (Farrell *et. al.*(2000); Amiraliyev *et. al.*(2010); Kadalbajoo *et. al.*(2010); Rao *et. al.*(2012)).

Amiraliyev *et. al.*(1999) proposed a fitted finite difference approximation on uniform mesh for IVP (1) which was first order convergent. Cen *et al.* proposed a hybrid scheme for IVP (1) on piecewise uniform Shishkin mesh. The difference equation has been solved for errors using Gronwall's inequality (Willett *et. al.* (1965)), they got almost second order convergence for the numerical solution and the scaled numerical derivatives.

The main goal is to construct a robust numerical scheme for the approximate solution of IVP (1) and its derivatives. Motivated from (Cen *et. al.*(2017)), a fitted mesh approach has been developed to handle the IVP (1). Taking different behavior of the solution in to account, different types of layer-adapted meshes like Shishkin mesh (S mesh) and Bakhvalov-Shishkin mesh (B-S mesh) are constructed. Afterwards, we develop a hybrid scheme by combining the cubic spline approximation on the inner layer with the midpoint upwind approximation on the smooth layer (LinB (2009)). Since estimation for numerical derivatives are desirable in many applied field (Mohapatra *et. al.*(2009); Priyadharshin *et. al.* (2009); Zheng *et. al.*(2015)), we give error bound for the numerical derivatives. Here, denotes a generic positive constant independent of and the mesh parameter.

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II. PROPERTIES OF THE SOLUTION AND ITS DERIVATIVES

Lemma 2.1 The solution $y(x)$ and the derivatives of IVP (1) satisfies the following bounds:

$$\left\|y^{l}(x)\right\| \le C\varepsilon^{-l}\left\{\phi+\max_{0\le t\le x}|g(t)|+\varepsilon^{j}\max_{0\le t\le x}|g(t)|+\int_0^x |g'(t)|dt\right\} \tag{2}$$

for all $x\in\overline{\Omega},\quad 0\le l<6$, where $\phi=\sqrt{\left|\eta^2+q(0)\lambda^2-2g(0)\right|},\ j=\max\{0,l-2\}$.

Proof. The idea of the proof is given in [5].

We decompose the solution $y(x)$ of the IVP (1) into smooth and layer components as:

$y(x)=u(x)+v(x)$. The smooth part $u(x)$ is satisfies the following sets of problem:

$$\begin{cases} Lu(x)=g(x) \\ u(0)=u_0(0)+\varepsilon u_1(0)+\varepsilon^2u_2(0)++\varepsilon^3u_3(0) \\ u'(0)=u'_0(0)+\varepsilon u'_1(0)+\varepsilon^2u'_2(0)+\varepsilon^3u'_3(0) \end{cases} \tag{3}$$

Here, u_0,u_1,u_2,u_3 respectively, the solution of the following problems:

$$\begin{cases} u_0(x)=\dfrac{g(x)}{q(x)}, \\ u_1(x)=\dfrac{1}{q(x)}\{\varepsilon u_0''-p(x)u_0'(x)\}, \\ u_2(x)=\dfrac{1}{q(x)}\{\varepsilon u_1''-p(x)u_1'(x)\}, \\ u_3(x)=\dfrac{1}{q(x)}\{\varepsilon u_2''-p(x)u_2'(x)\}. \end{cases} \tag{4}$$

The singular components $v(x)$ is satisfying the following problem:

$$\begin{cases} Lv(x)=0, \\ v(0)=\lambda-u(0), \\ v'(0)=\dfrac{\eta}{\varepsilon}-u'(0). \end{cases} \tag{5}$$

Lemma 2.2 The derivatives of smooth components $u^{l}(x)$ satisfy the following:

$$\left\|u^{l}(x)\right\|\le C,\ 0\le l\le 4. \tag{6}$$

Proof. One can find a similar kind of proof in [6].

Lemma 2.3 The singular components $v(x)$ and the derivatives $v^{l}(x)$ have the following estimates:

$$\left\|v^{l}(x)\right\|\le C(\varepsilon^{3-l}+\varepsilon^{-l}e^{-mx/\varepsilon}+\varepsilon^{1-l}e^{-mx/2\varepsilon}),\ 0\le l\le 5 \tag{7}$$

where

$$m=\begin{cases} (p(0)-\sqrt{(p^2(0)-4q(0))})/2, & p^2(0)>4q(0), \\ p(0)/4, & p^2(0)=4q(0), \\ p(0)/2, & p^2(0)<4q(0). \end{cases} \tag{8}$$

Proof. The idea of the proof is given in (Cen *et. al.*).

2.1. Cubic spline approximation

Consider the cubic spline approximation on a variable mesh $\Omega^N=\{0=x_0<x_1<......x_N=1\}$, and let $h_i=x_i-x_{i-1}$. for given values $Y(x_0),Y(x_1),......Y(x_N)$ of a function $y(x)$ on Ω^N, there exists a cubic spline function, $R(x)$ with following assumptions:

(i) $R(x)$ agree with a third degree polynomial on each intervals $[x_{i-1},x_i]$, $i=1,.......N$.

(ii) $R(x)\in C^2(\overline{\Omega})$.

(iii) $R(x_i)=Y(x_i)$ for $i=1,.......N$.

Now for $x\in[x_{i-1},x_i]$, the cubic spline function is defined as follows:

$$R(x)=\frac{(x_i-x)^3}{6h_i}S_{i-1}+\frac{(x-x_{i-1})^3}{6h_i}S_i +\left(Y(x_{i-1})-\frac{h_i^2}{6}S_{i-1}\right)\left(\frac{(x_i-x)}{6h_i}\right) +\left(Y(x_i)-\frac{h_i^2}{6}S_i\right)\left(\frac{(x-x_{i-1})}{6h_i}\right)$$

where $S_i=R''(x_i)$. Now from the elementary properties of the spline, we have,

$$\frac{h_i}{6}S_{i-1}+\frac{h_i+h_{i+1}}{6}S_i+\frac{h_{i+1}}{6}S_{i+1}=\frac{Y(x_{i+1})-Y(x_i)}{h_{i+1}}-\frac{Y(x_i)-Y(x_{i-1})}{h_i}. \tag{9}$$

For obtaining second order approximation for the first derivative of $y(x)$ one can refer (Priyadharshin *et. al.* (2010)). Now for the IVP (1) consider the approximation:

$$\varepsilon^2 S_j + p(x_j)Y'(x_j) + q(x_j)Y(x_j) = g(x_j),$$

for $j = i, i \pm 1.$ (10)

Substituting this in (9), we develop the linear systems, for $i = 1,.......N-1,$

$$L_{Cu}Y_i = r^- Y(x_{i-1}) + r^c Y(x_i) + r^+ Y(x_{i+1}) = G(x_i),$$ (11)

where
$$\begin{cases} r^- = \dfrac{h_i}{6\varepsilon^2} q_{i-1} + \dfrac{1}{h_i} - \dfrac{(h_i + 2h_{i+1})}{6\varepsilon(h_i + h_{i+1})} p_{i-1} - \dfrac{h_{i+1}}{3\varepsilon h_i} p_i + \dfrac{h_{i+1}^2}{3\varepsilon h_i} p_{i+1}, \\ r^c = \dfrac{(h_i + h_{i+1})}{3\varepsilon^2} q_i - \dfrac{(h_i + h_{i+1})}{h_i h_{i+1}} + \dfrac{(h_i + h_{i+1})}{6\varepsilon h_{i+1}} p_{i-1} + \dfrac{(h_{i+1}^2 - h_i^2)}{3\varepsilon h_i h_{i+1}} p_i - \dfrac{(h_{i+1} + h_i)}{6\varepsilon h_i} p_{i+1}, \\ r^+ = \dfrac{h_{i+!}}{6\varepsilon^2} q_{i-1} + \dfrac{1}{h_{i+1}} - \dfrac{h_i^2}{6\varepsilon h_{i+1}(h_i + h_{i+1})} p_{i-1} + \dfrac{h_i}{3\varepsilon h_{i+!}} p_i + \dfrac{(h_{i+1} + h_i)}{6\varepsilon h_i(h_i + h_{i+1})} p_{i+1}, \end{cases}$$

and $G(x_i) = \dfrac{h_i}{2} g(x_{i-1}) + (h_i + h_{i+1}) g(x_{i-1}) + \dfrac{h_{i+1}}{2} g(x_{i+1}).$

III. THE DIFFERENCE SCHEME

To approximate IVP (1), we introduce a hybrid scheme on a non-uniform mesh of N intervals Ω^N. Now, define a transition parameter $\sigma = \min\left\{\dfrac{T}{2}, \dfrac{4}{m}\ln N\right\}$, which divide Ω^N into two subdomains with $N/2$ equal subintervals. Define a mesh generating function ζ with $\zeta(0) = 0$ and $\zeta(1/2) = \ln N$ .Then, the nodal points are given by

$$x_i = \begin{cases} \dfrac{2\varepsilon}{\beta}\zeta(t_i), & i = 0,1,..............N/2, \\ 1 - \left(1 - \dfrac{2\varepsilon}{\beta}\ln N\right)\left(\dfrac{2(N-i)}{N}\right), & i = N/2+1,..........N. \end{cases}$$ (12)

where, $t_i = i/N.$ Let define the function χ, that is related to ζ with $\zeta = -\ln \chi$, and satisfies $\chi(0) = 1$ and $\chi(1/2) = N^{-1}$. Then the characterizing function χ is given by

$$\chi(t) = \begin{cases} e^{-2(\ln N)t}, & \text{(S mesh)}, \\ 1 - 2(1 - N^{-1})t, & \text{(B-S mesh)}. \end{cases}$$

3.1 Hybrid difference scheme

In this scheme, we use the cubic spline approximation define in (11) in the fine mesh region and the midpoint upwind approximations on the coarse mesh region of Ω^N.

Let
$$L_{Mp}Y_i = \frac{2\varepsilon^2}{h_i + h_{i+1}}\left(\frac{Y_{i+1} - Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i}\right) + p_{i-1/2}\left(\frac{Y_i - Y_{i-1}}{h_i}\right) + q_{i-1/2}\overline{Y}_i = g_{i-1/2}$$ (13)

where, $Y_i = Y(x_i)$, $\overline{Y}_i = (Y_{i-1} + Y_i)/2$ and $p_{i-1/2} = p((x_{i-1} + x_i)/2)$; similar for $q_{i-1/2}$ and $g_{i-1/2}$.

Thus, the hybrid finite approximation for IVP (1) takes of the form:

$$L_H Y_i = \begin{cases} L_{Cu}Y_i, & 1 \le i < .N/2, \\ L_{Mp}Y_i & N/2 \le i < N-1. \end{cases}$$ (14)

and

$$L_0 Y_0 = \frac{-Y_2 + 4Y_1 - Y_0}{-(x_2 - x_1) + 4(x_1 - x_0)} = \frac{\eta}{\varepsilon}, \qquad Y(0) = \lambda.$$ (15)

Proposition 3.1

Let y and Y_i be the solutions of IVP (1) and discrete problem (14 -15) respectively. Then, the parameter uniform estimate is given by:

$$\|y_i - Y_i\| \le C(N^{-1}\ln N)^2, \quad \text{(S mesh)}$$

$$\|y_i - Y_i\| \le CN^{-2}, \quad \text{(B-S mesh)}$$

Proposition 3.2

Let y and Y_i be the solutions of IVP (1) and discrete problem (14-15) respectively. Then, the error estimate for the scaled numerical derivatives is given as:

$$\varepsilon\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|\le C(N^{-1}\ln N)^2$$, (S mesh)

$$\varepsilon\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|\le CN^{-2}$$, (B-S mesh)

IV. NUMERICAL RESULTS AND DISCUSSIONS

Example 4.1 Consider the IVP:

$$\varepsilon^2 y''(x)+\varepsilon(3+x\sin(x))y'(x)$$
$$+(1+e^x)y(x)=g(x),\quad 0<x\le 1,$$

$$y(0)=2, y'(0)=1-\frac{1}{2\varepsilon},$$

(16)

where $g(x)$ is taken in such a way that $y(x)=1+x-x^2+e^{-x/2\varepsilon}$ is exact solution . The error of the difference approximation is computed in the discrete maximum norm. The maximum point wise errors E^N and the scaled errors of numerical derivatives D^N are defined by:

$$E^N=\max_{1\le i\le N}\left|y(x_i)-Y_i\right|,$$

$$D^N=\max_{1\le i\le N}\left|\frac{Y_{i+1}-Y_i}{h_{i+1}}-y'_{i+1/2}\right|,$$

where U_i is obtained by the proposed methods. Then the corresponding rate of convergence are given by:

$$r_E=\frac{\ln E^N-\ln E^{2N}}{\ln(2\ln N/\ln(2N))},\quad r_D=\frac{\ln D^N-\ln D^{2N}}{\ln(2\ln N/\ln(2N))}.$$

Table 1 represents the maximum pointwise error and the corresponding rate for $\varepsilon=1e-4$ and $\varepsilon=1e-8$. Similarly, Table 2 displays the scaled error and the corresponding rate r_D. The results clearly indicate that the present scheme is uniformly convergence and of quadratic order. For proper visualization of the rate of a convergence, the log log plots of the E^N and D^N are shown in Figure 1.

A singularly perturbed second order IVP is taken into consideration. We presented the hybrid scheme on Shishkin type meshes (S mesh and B-S mesh). The scheme proposed in this article is a combination of cubic spline approximation on the singular region and the midpoint upwind approximation on the smooth region. The proposed method generates a second order ε-uniform convergence rate of the numerical solution and the scaled numerical derivatives. The efficacy of the proposed scheme can be easily seen from the numerical results and the approximation coincides with the theoretical results.

Conflict of interests

The authors do not have any conflict of interest regarding the publication of the paper.

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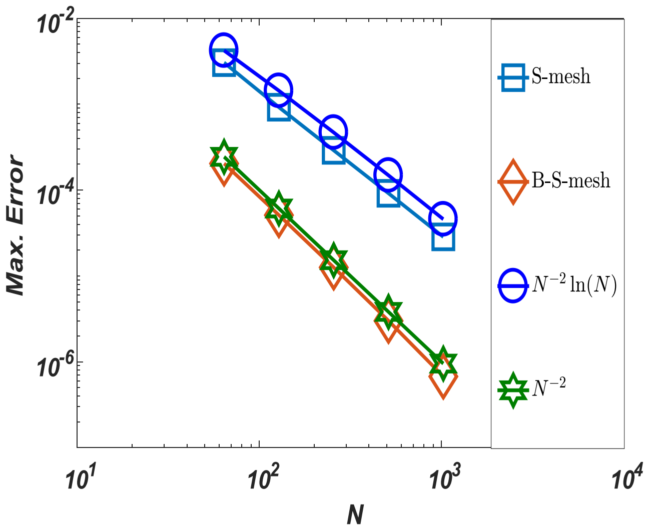
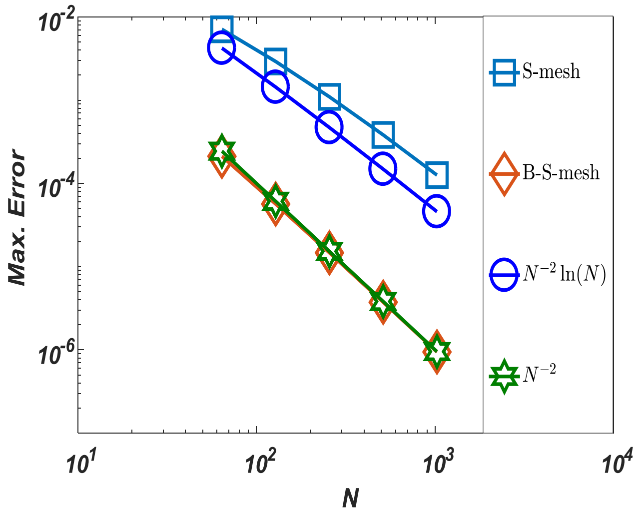


(a) Maximum error



(b) Maximum scaled error

Figure 1: Loglog plot for Example 4.1

Table 1: E^N and r_E generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
332	1.0797e-2 2.4748	1.0797e-2 2.4748	7.9889e-4 2.6732	8.0394e-4 2.6629
664	3.0497e-3 2.2184	3.0497e-3 2.2184	2.0391e-4 2.5678	2.0617e-4 2.5474
1128	9.2248e-4 2.0653	9.2249e-4 2.0665	5.1092e-5 2.5055	5.2215e-5 2.4657
2256	2.9041e-4 2.0031	2.9022e-4 2.0019	1.2572e-5 2.4877	1.3138e-5 2.4038
5512	9.1725e-5 1.9885	9.1726e-5 2.0359	3.0046e-6 2.5416	3.2953e-6 2.3558

Table 2: D^N and r_D generated by the hybrid scheme for Example 4.1.

N	S mesh		B-S mesh	
	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
332	1.5617e-2 1.5422	1.5617e-2 1.5422	7.6177e-4 2.5062	7.6177e-4 2.5062
664	7.1034e-3 1.6511	7.1034e-3 1.6511	2.1174e-4 2.4513	2.1175e-4 2.4503
1128	2.9172e-3 1.7480	2.9173e-3 1.7479	5.6525e-5 2.4115	5.6525e-5 2.4114
2256	1.0968e-3 1.8299	1.0969e-3 1.8300	1.4661e-5 2.3754	1.4661e-5 2.3754
5512	3.8273e-4 1.8927	3.8273e-4 1.8920	3.7377e-6 2.3413	3.7378e-6 2.3409

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