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Galerkin Method for Solving Twelfth Order BVP's

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Abstract: *We deployed a numerical method to solve a twelfth order BVP's in terms of finite element Galerkin approach with the sextic B-splines as basis functions. The basis functions have been redefined into another set of basis functions, governing approximate solution satisfies given boundary conditions. To know the efficiency of the proposed numerical method, we have been examined the numerical scheme by applying this scheme on some twelfth order linear and nonlinear BVP's and these results compare with the exact solution available in the literature.*

Keywords: Galerkin method, Sextic B-spline, Basis function, Twelfth order BVP's.

I. INTRODUCTION

A general form of linear twelfth order BVP is

$$p_0(t)z^{(12)}(t)+p_1(t)z^{(11)}(t)+p_2(t)z^{(10)}(t)+p_3(t)z^{(9)}(t)+p_4(t)z^{(8)}(t)+p_5(t)z^{(7)}(t)$$
$$+p_6(t)z^{(6)}(t)+p_7(t)z^{(5)}(t)+p_8(t)z^{(4)}(t)+p_9(t)z'''(t)+p_{10}(t)z''(t)+p_{11}(t)z'(t)$$
$$+p_{12}(t)z(t)=q(t), \qquad a<t<b \tag{1}$$

subject to boundary conditions

$$z(a)=A_0,\ z(b)=C_0,\ z'(a)=A_1,\ z'(b)=C_1,\ z''(a)=A_2,\ z''(b)=C_2,$$
$$z'''(a)=A_3,\ z'''(b)=C_3,\ z^{(4)}(a)=A_4,\ z^{(4)}(b)=C_4,\ z^{(5)}(a)=A_4,\ z^{(5)}(b)=C_4. \tag{2}$$

where A_i's, C_i's are finite real constants, $p_i(t)$'s and $q(t)\in C([a,b])$.

Practically, these kinds of BVP's occur in the areas of hydrodynamics and hydromagnetic stability and other parts of applied sciences (Mathematics, Physics,…*etc*) and engineering. The book by Chandrasekhar(1981) explained in details, how it is modeled by twelfth order BVP's. In Agarwal (1986) book contains theorems on the existence and uniqueness of these types of problems and its solutions without any numerical approximation.

In general, solving these kinds of problems for the analytical solution is very difficult and one can solve these problems in some special or rare cases only. From a few years, many researchers have been developed numerical schemes to get the approximate solutions to these kinds of problems. To mention a few, using finite difference (Twizell *et. al.,* 1991; Twizell and Boutayeb, 1994 ; Dijidejeli and Twizell, 1993) developed numerical schemes to solve higher even order differential equations, eigenvalue problems arising in the thermal variability. Using spline twelve and thirteen-degree polynomials, Twizell and Siddiqi (1997, 2006) developed a numerical scheme using finite difference method for eqs. (1)-(2). Some of the numerical methods and semi-analytical methods developed to solve these kinds of problems and some of them are Differential transform method (Siraj-Ul Islam *et. al.,* 2009), Adomain decomposition method (A. M. Wazwaz, 2009), homotopy-perurbation method (Ravikanth and Aruna, 2009), Variational iteration method (Ravikanth and Aruna, 2009; Aslam Noor, and Mohyud-Din, 2008).

In the present paper, we aim is to develop a numerical algorithm to solve the BVP mentioned in the above form eqs. (1) - (2) with the support of sextic B-splines by the Galerkin method. To solve a huge number of diversity of applications of both linear and nonlinear BVP's, the proposed method has been applied. The nonlinear problems have been solved with the help of the quasilinearization technique (Kalaba and Bellman, 1965). The books (Bers *et. al.*, 1964; Lions and Magenes, 1972; Mitchel and Wait, 1977) consisting of validation of the Galerkin method.

II. METHOD OF PROCEDURE

The sextic B-splines are defined in (Prenter, 1989; Carl de-Boor, 2001; Schoenberg, 1966). We defined the approximate solution of the equation eqs. (1) - (2) is *z(t)* as

$$z(t)=\sum_{j=-3}^{n+2}\alpha_j B_j(t) \tag{3}$$

where α_j's are the control points(nodal parameters), $B_j(t)$'s are sextic B-splines and these are piecewise $C^5[a, b]$ functions on given domain and these are forms a basis for spline polynomial space $S_6([a, b])$. Hence *z(t)* is also piecewise $C^5[a, b]$ function on domain. If the given boundary conditions eq. (2) are fulfilled by the approximate solution then numerical scheme gives more precise results. Thus, by making use of the sextic basis functions and boundary conditions we will get a new set of sextic basis functions. These functions will satisfy boundary conditions. The remodified basis functions are obtained from the following procedure.

We got the following equations by approximating boundary conditions mentioned in eq. (2) with the help of the approximation solution mentioned in eq. (3)

$$A_0 = z(a) = z(t_0) = \sum_{j=-3}^{2}\alpha_j B_j(t_0),$$
$$C_0 = z(b) = z(t_n) = \sum_{j=n-3}^{n+2}\alpha_j B_j(t_n) \tag{4}$$

$$A_1 = z'(a) = z'(t_0) = \sum_{j=-2}^{2}\alpha_j B_j'(t_0),$$
$$C_1 = z'(b) = z'(t_n) = \sum_{j=n-3}^{n+1}\alpha_j B_j'(t_n) \tag{5}$$

$$A_2 = z''(a) = z''(t_0) = \sum_{j=-1}^{2}\alpha_j B_j''(t_0),$$
$$C_2 = z''(b) = z''(t_n) = \sum_{j=n-3}^{n}\alpha_j B_j''(t_n) \tag{6}$$

$$A_3 = z'''(a) = z'''(t_0) = \sum_{j=0}^{2}\alpha_j B_j'''(t_0),$$
$$C_3 = z'''(b) = z'''(t_n) = \sum_{j=n-3}^{n-1}\alpha_j B_j'''(t_n) \tag{7}$$

$$A_4 = z^{(4)}(a) = z^{(4)}(t_0) = \sum_{j=1}^{2}\alpha_j B_j^{(4)}(t_0),$$
$$C_4 = z^{(4)}(b) = z^{(4)}(t_n) = \sum_{j=n-3}^{n-2}\alpha_j B_j^{(4)}(t_n) \tag{8}$$

$$A_5 = z^{(5)}(a) = z^{(5)}(t_0) = \alpha_2 B_2^{(5)}(t_0),$$
$$C_5 = z^{(5)}(b) = z^{(5)}(t_n) = \alpha_{n-3} B_{n-3}^{(5)}(t_n) \tag{9}$$

removing α_{-3} to α_2 and α_{n-3} to α_{n+2} from the above equations eqs. (3) - (9), we obtain the approximate solution *z(t)* as

$$z(t) = wt(t) + \sum_{j=3}^{n-4}\alpha_j \tilde{B}_j(t) \tag{10}$$

where

$$wt(t) = wt_5(t) + \frac{A_5 - wt_5^{(5)}(t_0)}{U_2^{(5)}(t_0)}U_2(t) + \frac{C_5 - wt_5^{(5)}(t_n)}{U_{n-3}^{(4)}(t_n)}U_{n-3}(t)$$

$$wt_5(t) = wt_4(t) + \frac{A_4 - wt_4^{(4)}(t_0)}{S_1^{(4)}(t_0)}S_1(t) + \frac{C_4 - wt_4^{(4)}(t_n)}{S_{n-2}^{(4)}(t_n)}S_{n-2}(t)$$

$$wt_4(t) = wt_3(t) + \frac{A_3 - wt_3'''(t_0)}{R_0'''(t_0)}R_0(t) + \frac{C_3 - wt_3'''(t_n)}{R_{n-1}'''(t_n)}R_{n-1}(t)$$

$$wt_3(t) = wt_2(t) + \frac{A_2 - wt_2''(t_0)}{Q_{-1}''(t_0)}Q_{-1}(t) + \frac{C_2 - wt_2''(t_n)}{Q_n''(t_n)}Q_n(t)$$

$$wt_2(t)=wt_1(t)+\frac{A_1-wt_1'(t_0)}{P_{-2}'(t_0)}P_{-2}(t)+\frac{C_1-wt_1'(t_n)}{P_{n+1}'(t_n)}P_{n+1}(t),\quad wt_1(t)=\frac{A_0}{B_{-3}(t_0)}B_{-3}(t)+\frac{C_0}{B_{n+2}(t_n)}B_{n+2}(t)$$

and

$$\tilde{B}_j(t)=U_j(t),\quad U_j(t)=\begin{cases} S_j(t)-\dfrac{S_j^{(4)}(t_0)}{S_1^{(4)}(t_0)}S_1(t), & j=2 \\ S_j(t), & j=3,\dots,n-4 \\ S_j(t)-\dfrac{S_j^{(4)}(t_n)}{S_{n-2}^{(4)}(t_n)}S_{n-2}(t), & j=n-3. \end{cases}$$

$$S_j(t)=\begin{cases} R_j(t)-\dfrac{R_j'''(t_0)}{R_0'''(t_0)}R_0(t), & j=1,2 \\ R_j(t), & j=3,\dots,n-4 \\ R_j(t)-\dfrac{R_j'''(t_n)}{R_{n-1}'''(t_n)}R_{n-1}(t), & j=n-3,n-2. \end{cases},\quad R_j(t)=\begin{cases} Q_j(t)-\dfrac{Q_j''(t_0)}{Q_{-1}''(t_0)}Q_{-1}(t), & j=0,1,2 \\ Q_j(t), & j=3,\dots,n-4 \\ Q_j(t)-\dfrac{Q_j''(t_n)}{Q_n''(t_n)}Q_n(t), & j=n-3,n-2,n-1. \end{cases}$$

$$Q_j(t)=\begin{cases} P_j(t)-\dfrac{P_j'(t_0)}{P_{-2}'(t_0)}P_{-2}(t), & j=-1,0,1,2 \\ P_j(t), & j=3,\dots,n-4 \\ P_j(t)-\dfrac{P_j'(t_n)}{P_{n+1}'(t_n)}P_{n+1}(t), & j=n-3,n-2,n-1,n. \end{cases},\quad P_j(t)=\begin{cases} B_j(t)-\dfrac{B_j(t_0)}{B_{-3}(t_0)}B_{-3}(t), & j=-2,-1,0,1,2 \\ B_j(t), & j=3,\dots,n-4 \\ B_j(t)-\dfrac{B_j(t_n)}{B_{n+2}(t_n)}B_{n+2}(t), & j=n-3,n-2,n-1,n,n+1. \end{cases}$$

Now, $\{\tilde{B}_j(t),\ j=3,\dots,n-4\}$ is the new-fangled basis to approximate solution space $S_6([a,b])$ and with these basis functions and implementing the Galerkin method to (1), we get

$$\int_{t_0}^{t_n}\Big[p_0(t)z^{(12)}(t)+p_1(t)z^{(11)}(t)+p_2(t)z^{(10)}(t)+p_3(t)z^{(9)}(t)+p_4(t)z^{(8)}(t)+p_5(t)z^{(7)}(t)$$
$$+p_6(t)z^{(6)}(t)+p_7(t)z^{(5)}(t)+p_8(t)z^{(4)}(t)+p_9(t)z'''(t)+p_{10}(t)z''(t)+p_{11}(t)z'(t)$$
$$+p_{12}(t)z(t)\Big]\tilde{B}_i(t)\,dt=\int_{t_0}^{t_n}q(t)\tilde{B}_i(t)dt \quad \text{for} \quad \mathrm{i}=3,4,\dots,\ \mathrm{n}-4. \tag{11}$$

Applying integrating by parts to the first 7-terms (higher order derivative terms) of the above integral eq. (11) and after applying conditions mentions in eq. (2), we obtained the system of linear equations and these were arranged in the following form.

$$\mathbf{A}\alpha=\mathbf{B} \tag{12}$$

where $\mathbf{A}=[a_{ij}]$;

$$a_{ij}=\int_{t_0}^{t_n}\{\frac{d^6}{dt^6}(p_0(t)\tilde{B}_i(t))\tilde{B}_j^{(6)}(t)+[\frac{d^6}{dt^6}(p_1(t)\tilde{B}_i(t))-\frac{d^5}{dt^5}(p_2(t)\tilde{B}_i(t))+\frac{d^4}{dt^4}(p_3(t)\tilde{B}_i(t))$$
$$-\frac{d^3}{dt^3}(p_4(t)\tilde{B}_i(t))+\frac{d^2}{dt^2}(p_5(t)\tilde{B}_i(t))-\frac{d}{dt}(p_6(t)\tilde{B}_i(t))+p_7(t)\tilde{B}_i(t)]\tilde{B}_j^{(5)}(t)$$
$$+p_8(t)\tilde{B}_i(t)\tilde{B}_j^{(4)}(t)+p_9(t)\tilde{B}_i(t)\tilde{B}_j'''(t)+p_{10}(t)\tilde{B}_i(t)\tilde{B}_j''(t)\ +p_{11}(t)\tilde{B}_i(t)\tilde{B}_j'(t)$$
$$+p_{12}(t)\tilde{B}_i(t))\tilde{B}_j(t)\}dt \quad \text{for} \quad \mathrm{i}=3,4,\ldots,\ \mathrm{n}-4;\ \mathrm{j}=3,4,\ldots,\mathrm{n}-4. \tag{13}$$

$\mathbf{B}=[b_i];$

$$b_i=\int_{t_0}^{t_n}\{q(t)\tilde{B}_i(t)-\frac{d^6}{dt^6}(p_0(t)\tilde{B}_i(t))wt^{(6)}(t)+[-\frac{d^6}{dt^6}(p_1(t)\tilde{B}_i(t))+\frac{d^5}{dt^5}(p_2(t)\tilde{B}_i(t))$$
$$-\frac{d^4}{dt^4}(p_3(t)\tilde{B}_i(t))+\frac{d^3}{dt^3}(p_4(t)\tilde{B}_i(t))-\frac{d^2}{dt^2}(p_5(t)\tilde{B}_i(t))+\frac{d}{dt}(p_6(t)\tilde{B}_i(t))$$
$$-p_7(t)\tilde{B}_i(t)]wt^{(5)}(t)-p_8(t)\tilde{B}_i(t)wt^{(4)}(t)-p_9(t)\tilde{B}_i(t)wt'''(t)-p_{10}(t)\tilde{B}_i(t)wt''(t)$$
$$-p_{11}(t)\tilde{B}_i(t)wt'(t)-p_{12}(t)\tilde{B}_i(t))wt(t)\}dt, \quad \text{for} \quad \mathrm{i}=3,4,\ldots,\ \mathrm{n}-4. \tag{14}$$

and $\alpha=[\alpha_3\ \alpha_4\ \ldots\alpha_{n-4}]^T$.

III. SOLUTION

The matrix $\mathbf{A}$ consisting integral element is in the form $\sum_{k=0}^{n-1}I_k$, where $I_k=\int_{t_k}^{t_{k+1}}w_i(t)w_j(t)X(t)dt$ and $w_i(t),\ w_j(t)$ are the new-fangled basis functions or their derivatives. The integral element $I_k=0$ if $(t_{i-3},t_{i+4})\cap(t_{j-3},t_{j+4})\cap(t_k,t_{k+1})=\varnothing$. To solve integral element I_k in the above form, we applied quadrature rule of Gauss-Legendre with seven point. Thus, we get thirteen diagonal band stiffness matrix $\mathbf{A}$. We solve obtain system $\mathbf{A}\alpha=\mathbf{B}$ for unknown control points (nodal parameters) $\alpha=[\alpha_3\ \alpha_4\ \ldots\alpha_{n-4}]^T$ with help of band matrix solution package.

IV. NUMERICAL RESULTS

We are solving the twelfth order BVP by the proposed numerical method to know its efficiency. In particular, we have applied the proposed numerical method on linear and nonlinear BVP's. The obtained approximate solutions by the proposed method compared with the exact solutions and these results presented in the tabular forms. We define max norm for computing error e to the each example and it is defined as $\|e\|_\infty=\underset{1\le i\le n}{Max}|e_i|$, where absolutely error (A.E) given by $|e_i|=|Y_i-y_i|$, here Y_i, y_i are exact and numerical solutions respectively at knot (grid point) $x=x_i$.

Example 1: The linear BVP is given by

$$z^{(12)}-z''+tz=-(120+20t-t^2+t^3)e^t,\quad 0<t<1 \tag{15}$$

subject to $z(0)=z(1)=0$, $z'(0)=1,\ z'(1)=-e,$
$z''(0)=0,\quad z''(1)=-4e,$
$z'''(0)=-3,\quad z'''(1)=-9e,$
$z^{(4)}(0)=-8,\ z^{(4)}(1)=-16e,$
$z^{(5)}(0)=-15,\ z^{(5)}(1)=-25e.$

The exact solution of the above differential equation is z $=t\,(1-t)\,e^t$.

To know the efficiency of this numerical scheme, the given domain is parted into ten equal subdomains by taking step size h=1/10 and applied on it. In the following Table 1 presented obtained numerical results to this problem eq. (15) and $1.800060\text{x}10^{-5}$ is the error in the above example by the proposed method.

Example 2: The nonlinear BVP is given by

$$z^{(12)}=11![e^{-12z}-\frac{2}{(1+t)^{12}}],\qquad 0\le t\le e^{\frac{1}{3}}-1$$

(16)

subject to $z(0)=0,\ z(e^{\frac{1}{3}}-1)=\frac{1}{3},\quad z'(0)=1,$
$z'(e^{\frac{1}{3}}-1)=e^{\frac{-1}{3}},\ z''(0)=-1,\ z''(e^{\frac{1}{3}}-1)=-e^{\frac{-2}{3}},$

$z'''(0)=2, \quad z'''(e^{\frac{1}{3}}-1)=2e^{-1}, \quad z^{(4)}(0)=-6,$
$z^{(4)}(e^{\frac{1}{3}}-1)=-6e^{\frac{-4}{3}}, \quad z^{(5)}(0)=24,$
$z^{(5)}(e^{\frac{1}{3}}-1)=24e^{\frac{-5}{3}}.$

and *z(t)=ln(1+t)* is the solution of the (16).

The quasilinearization technique [13] generates the sequence of linear BVP's to the nonlinear BVP(16) as

$$z^{(12)}_{(n+1)}-12\times 11!e^{-12z_{(n)}}z_{(n+1)}=11![e^{-12z_{(n)}}(1-12z_{(n)})-\frac{2}{(1+t)^{12}}] \qquad n=0,1,2,3,... \tag{17}$$

subject to $z_{(n+1)}(0)=0, \; z_{(n+1)}(e^{\frac{1}{3}}-1)=\frac{1}{3}, \; z'_{(n+1)}(0)=1, \; z'_{(n+1)}(e^{\frac{1}{3}}-1)=e^{\frac{-1}{3}},$

$$z''_{(n+1)}(0)=-1, \; z''_{(n+1)}(e^{\frac{1}{3}}-1)=-e^{\frac{-2}{3}}, \; z'''_{(n+1)}(0)=2, \; z'''_{(n+1)}(e^{\frac{1}{3}}-1)=2e^{-1},$$

$$z^{(4)}{}_{(n+1)}(0)=-6, \; z^{(4)}{}_{(n+1)}(e^{\frac{1}{3}}-1)=-6e^{\frac{-4}{3}}, \; z^{(5)}{}_{(n+1)}(0)=24, \; z^{(5)}{}_{(n+1)}(e^{\frac{1}{3}}-1)=24e^{\frac{-5}{3}}.$$

here $(n+1)^{th}$ approximation for $z(t)$ is $z_{(n+1)}$. The given domain is parted into ten equal subdomains by taking step size h=1/10 and we have been solved a set of linear problems eq. (17) in a sequence by the proposed method. The Table 2 present numerical results of eq. (16) which are obtained by this numerical method and $6.958842\text{x}10^{-6}$ is the error in the above example by the proposed numerical method.

Table 1: Numerical results for Example1

x	A.E.
0.1	3.278255E-07
0.2	6.407499E-07
0.3	5.364418E-07
0.4	7.092953E-06
0.5	1.800060E-05
0.6	1.427531E-05
0.7	7.092953E-06
0.8	2.652407E-06
0.9	8.940697E-06

Table 2: Numerical results for Example 2

X	A.E.
3.956125E-02	6.072223E-07
7.912249E-02	1.192093E-07
1.186837E-01	2.980232E-07
1.582450E-01	4.217029E-06
1.978062E-01	3.293157E-06
2.373675E-01	8.493662E-07
2.769287E-01	1.576543E-05
3.164900E-01	6.958842E-06
3.560512E-01	2.473593E-06

V. CONCLUSIONS

We are demonstrated introduced a numerical algorithm for solving twelfth order BVP's with help of sextic B-spline as functions in terms of the Galerkin approach and applied on a few problems. To get a more approximate solution, basis functions changed into another set of new basis functions and these are vanishes on the boundary where eqn (2) prescribed. The error in the numerical results which was obtained by the deployed numerical scheme is too small. The asset of the established numerical scheme is easy to solve applications, simple, accuracy, and efficiency.

Conflict of Interest
Both the authors have equal contribution in this work and it is declared that there is no conflict of interest for this publication.

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