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# Richardson extrapolation technique for singularly perturbed delay parabolic partial differential equation

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**Abstract:** *This article presents a higher-order parameter uniformly convergent method for a singularly perturbed delay parabolic reaction-diffusion initial-boundary-value problem. For the discretization of the time derivative, we use the Crank-Nicolson scheme on the uniform mesh and for the spatial discretization, we use the central difference scheme on the Shishkin mesh, which provides a second order convergence rate. To enhance the order of convergence, we apply the Richardson extrapolation technique. We prove that the proposed method converges uniformly with respect to the perturbation parameter and also attains almost fourth order convergence rate. Finally, to support the theoretical results, we present some numerical experiments by using the proposed method.*

**Keywords:** Singular perturbation, Delay parabolic problems, Crank-Nicolson scheme, Richardson extrapolation.

## I. INTRODUCTION

In this article, we consider the following singularly perturbed delay parabolic reaction-diffusion Initial-Boundary-Value Problems (IBVP):

$$\begin{cases} \left(\frac{\partial}{\partial t} + L_{\varepsilon,x}\right) u(x,t) = -b(x,t)u(x,t-\tau) + f(x,t), \\ \qquad (x,\ t) \in D, \\ u(x,t) = \theta_b(x,t), \quad (x,t) \in \Gamma_b, \\ u(0,t) = \theta_l(t) \quad on \ \ \Gamma_l = \{(0,t): 0 \le t \le T\}, \\ u(0,t) = \theta_l(t) \quad on \ \ \Gamma_l = \{(0,t): 0 \le t \le T\}, \end{cases} \tag{1.1}$$

where, $L_{\varepsilon,x} u(x,t) = -\varepsilon u_{xx}(x,t) + a(x)u(x,t)$.

Here $\Omega = (0,1), D = \Omega \times (0,T]$, $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. $\Gamma_b$ and $\Gamma_r$ are the left and the right sides of the rectangular domain $D$ corresponding to $x = 0$ and $x = 1$, respectively. $\Gamma_b = [0,1] \times [-\tau, 0]$. Also, $0 < \varepsilon \ll 1$ and $\tau > 0$ are given constants. The functions $a(x), b(x,t), f(x,t)$ on $D$ and $\theta_b(x,t)$, $\theta_l(t)$, $\theta r(t)$ on $\Gamma$, are sufficiently smooth, bounded functions that satisfy, $a(x) \ge \beta \ge 0, b(x,t) > 0$ on $D$. The terminal time $T$ is assumed to satisfy the condition $T = k\tau$ for some positive integer $k$. The required compatibility conditions at the corner points and the delay terms are $\theta_b(0,0) = \theta_l(0), \theta_b(1,0) = \theta r(0)$, and

$$\frac{d\theta_l(0)}{dt} - \varepsilon \frac{\partial^2 \theta_b(0,0)}{\partial x^2} + a(0)\theta_b(0,0) = -b(0,0)\theta_b(0,-\tau) + f(0,0),$$

$$\frac{d\theta_r(0)}{dt} - \varepsilon \frac{\partial^2 \theta_b(1,0)}{\partial x^2} + a(1)\theta_b(1,0) = -b(1,0)\theta_b(1,-\tau) + f(1,0).$$

Under the above assumptions and compatibility conditions, problem (1.1) admits a unique solution and the solution exhibits boundary layers along x=0, x=1(Ansari *et. al*.(2007); Kumar *and* Sekhara Rao (2010)). One can refer (Farrel *et. al*.(2000); Bansal *et. al*.(2015); Salama and Al-Amerya (2018); Govindarao and Mohapatra (2018)) reference therein for more details of singular perturbation.

There are few articles dealing with the theory and the numerical methods for equation (1.1). Ansari *et. al*.(Ansari *et. al*. (2007)) solved the singularly perturbed delay parabolic reaction diffusion problem on piecewise uniform Shishkin mesh. Das and Natesan. (Das and Natesan (2018)) solved the delay parabolic convection diffusion problem. But most of the methods discussed above using finite difference schemes are of first order or second order accurate. So there is a need of higher order accurate for (1.1).

Richardson extrapolation technique is one of post processing technique used to provide a approximate numerical solution and to increase the order of convergence. This technique is used by Mohapatra and Natesan (Mohapatra & Natesan (2008)) for solving singularly perturbed delay two point BVPs while Shishkin *et. al*. (Shishkin*et. al*. (2010)) applied this idea to solve the parabolic reaction-diffusion equation. The aim of this work is to provide a fourth order convergent solution for (1.1) using the Richardson extrapolation technique. First, we use the central difference scheme for the spatial direction on Shishkin mesh and the implicit Euler method for time direction on uniform mesh. Here, we solve the problem (1.1) with $N$ and $M$ number of subintervals in spatial and temporal direction respectively, after that we solve (1.1) with $2N$ and $4M$ number of subintervals. Then by combining these two solutions properly, we enhance the order of convergence from second order to fourth order in spatial direction and first order to second order in time direction.

# II. MATHEMATICAL MODEL AND COMPUTATIONAL MODEL

On time domain [0, T], we use uniform mesh with time step$\Delta t$,

$$\Omega_t^M = \{t_n = n\Delta t, n = 0.\dots M, t_M = T,\ \Delta t = T/M\},$$

$$\Omega_t^p = \{t_j = j\Delta t, j = 0.\dots p, t_p = \tau,\ \Delta t = \tau/p\},$$

where, $M$ is number of mesh points in t-direction on the interval [0, $T$] and $p$ is the number of mesh points in $[-\tau, 0]$. The step length $\Delta t$satisfies $p\Delta t = \tau$, where $p$ is $a$ positive integer,$t_n = \Delta t, n \geq -p$. To discretize the time variable for (1.1), we use the Crank-Nicolson method, which is given by

$$\begin{cases} u^{-j} = \theta_b(x, -t_j), \quad \text{for } j = 0,\dots,p, \quad x \in \overline{D}, \\ \left(I + \frac{\Delta t}{2} L_{\varepsilon,x}\right) u^{n+1} = \frac{\Delta t}{2}\left(-b^{n+1}u^{n-p+1} - b^n u^{n-p} + f^{n+1} + f^n\right) + \left(I - \frac{\Delta t}{2} L_{\varepsilon,x}\right) u^n, \\ u^{n+1}(0) = \theta_l(t_{n+1}),\ u^{n+1}(1) = \theta_r(t_{n+1}), \end{cases} \tag{2.1}$$

where, $f^n = f(x, t_n)$,$c^n = c(x, t_n)$, $u^n = u(x, t_n)$ is the semidiscrete approximation to the exactsolution $u(x,$ t) of (1.1) at the time level $t_n = \Delta t$.

# III. NUMERICAL APPROXIMATION

Here, we propose the numerical schemes to solve the IBVP (1.1). We discretize the IBVP (1.1) using the Crank-Nicolson scheme on a uniform mesh in time direction and the central difference scheme on a Shishkin mesh in the spatial direction. For the construction of the Shishkin mesh, one may refer (Miller *et. al*. (1996); Shishkin and Shishkina (2010)).

### 3.1 Spatial discretization

Let '$\sigma$' denotes a mesh transition parameter defined by$\sigma = min\left\{\frac{1}{4},\ \rho_0\sqrt{\varepsilon}lnN\right\}$, where$\rho_0 \geq \frac{2}{\beta}$. We divide the domain $\overline{\Omega} = [0, 1]$ into three sub-domains as $\overline{\Omega} = \overline{\Omega_l} \cup \overline{\Omega_c} \cup \overline{\Omega_r}$, where $\Omega_l = (0,\ \sigma]$, $\Omega_c = (\sigma,\ 1 - \sigma]$ and $\Omega_r = (1 - \sigma,\ 1]$. We assume that $N = 2^r$ with $r \geq 3$ is the total number of subintervals in the partitions of $[0, 1]$. We specify the mesh $\Omega_x = \{x_i \in (0, 1),\ i = 0,\dots,N\}$, where

$$x_i = \begin{cases} \dfrac{4i\sigma}{N}, & \text{for } i = 0,\dots,\dfrac{N}{4}, \\ \dfrac{2i(1-2\sigma)}{N}, & \text{for } i = \dfrac{N}{4} + 1,\dots,\dfrac{N}{4}, \\ \dfrac{4i\sigma}{N}, & \text{for } i = \dfrac{3N}{4} + 1,\dots,N. \end{cases}$$

We define the discretized domain $D^N = \Omega_x^N \times \Omega_t^M$on$D$, $\Gamma^N = \Omega_x^N \times \Omega_t^p$ $on$ $\Gamma$. Note that, whenever $\sigma = \frac{1}{4}$, the mesh is uniform and on the other hand when $\sigma = \rho_0\sqrt{\varepsilon}lnN$,the mesh is condensing near the boundaries $\Gamma_l$ and $\Gamma_r$, here$x_i - x_{i-1} = 4\sigma N^{-1}$. Consider the finite difference approximation for (1.1) on domain $\Omega_x^N$. Denote $h_j = x_j - x_{j-1}$. Given a mesh function $\phi_j$, the backward and the central difference operators as:

$D_x^- \phi_j^n = \frac{\phi_j^n - \phi_{j-1}^n}{h_j}$, $\quad D_x^+ D_x^- \phi_j^n = \frac{2}{h_j + h_{j+1}}\left(\frac{\phi_{j+1}^n - \phi_j^n}{h_{j+1}} - \right.$ $\phi jn - \phi j - 1nhj$.

Also define the backward difference operator in time by $D_t^- \phi_j^n = \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t}$, where $\phi_j^n = \phi(x_i, t_n)$.We propose the following the numerical scheme to solve IBVP (1.1), the Crank-Nicolson scheme for the time derivative, and the central difference scheme for the spatial derivatives, which is defined as:

$$2D_t^- U_i^{n+1} + L_\varepsilon U_i^{n+1} = -b_i^{n+1} U_i^{n-p+1} - b_i^n U_i^{n-p} + f_i^n + f_i^{n+1} - L_\varepsilon U_i^n, \tag{3.1}$$

here, $L_\varepsilon U_i^n = -\varepsilon D_x^+ D_x^- U_i^n + a_i U_i^n$,$f_i^n = f(x_i, t_n)$, $b_i^n = c(x_i, t_n)$, $a_i = a(x_i)$, for $i = 1, 2,\dots, N - 1$.

### 3.2 Fully discrete scheme

Using the scheme (2.1) and after rearranging the terms in (3.1), the fully discrete scheme obtained is given by,

$$\begin{cases} r_i^- U_{i-1}^{n+1} + r_i^o U_i^{n+1} + r_i^+ U_i^{n+1} = 2g_i^n, \\ U_0^{n+1} = \theta_l(t_{n+1}),\ U_0^{n+1} = \theta_r(t_{n+1}), \\ U_0^{-j} = \theta_l(x_i, t_{n+1}), \text{for } j = 0,\dots,p, \text{and } i = 1,\dots,N-1. \end{cases} \tag{3.2}$$

Here,

$$\begin{cases} r_i^- = \Delta t\left(-\dfrac{2\varepsilon}{\hat{h}_i \mathrm{h}_i}\right), \\ r_i^o = \Delta t\left(-\dfrac{2\varepsilon}{\hat{h}_i \mathrm{h}_i} + b_i^{n+1}\right) + 1, \\ r_i^+ = \Delta t\left(-\dfrac{2\varepsilon}{\hat{h}_i \mathrm{h}_i}\right), \end{cases}$$

for $0 < i \le N - 1$, $g_i^n = \frac{\Delta t}{2}\left(-b_i^{n+1} U_i^{n-p+1} -\right.$ *binUin−p+fin+fin+1+1−Δt2Uin.*

The difference equations (3.2), at each time level n+1 form a tri-diagonal system of $N-1$equations with $N-1$ unknowns. The tri-diagonal systems have the following properties:

$r_i^- < 0, r_i^o > 0,\ r_i^+ < 0$, for $i = 1, \dots, N-1$.

These matrixes have the diagonal predominance with respect to columns. Therefore, to solve the tri-diagonal system, we use Thomas algorithm. For a brief detail of Thomas algorithm and stability one can refer (Mohapatra and Natesan (2008); Kumar and Sekhara Rao (2010); Raji Reddy & Mohapatra (2015)).

**Theorem 1**

Let $u$and $U$be the solutions of (1.1) and (3.1) respectively, satisfying the compatibility conditions. Then, the error of the finite difference scheme (3.1) satisfies the following estimate

$\max_{i,n}|(u-U)(x_i,\ t_n)| \le C((N^{-1}\ln N)^2 + \Delta t^2)$,
for$i = 1, \dots, N-1$, where

$U(x_i,\ t_n) = U_i^n$, for $(x_i, t_n) \in D^N$.

## IV. RICHARDSON EXTRAPOLATION TECHNIQUE

To increase the accuracy of the numerical solutions of the scheme, we use the Richardson extrapolation technique. To apply the technique, we solve the discrete problem (3.1) on the fine mesh $D^{2N} = \Omega_x^{2N} \times \Omega_t^{4M}$. with $2N$ mesh intervals in the spatial direction and $4M$ mesh intervals in the time direction, where $\overline{\Omega}_x^{2N}$ is a piecewise uniform Shishkin mesh having the same transitions points as $\overline{\Omega}_x^N$ and obtained by bisecting each mesh interval of$\overline{\Omega}_x^N$.
Clearly,
$D^N = \{(x_i, t_n)\} \subset D^{2N} = \{(\bar{x}_i, \bar{t}_n)\}$. Therefore, the corresponding mesh the mesh is$\overline{\Omega}_x^N = \{\bar{x}_i \in (0,1),\ i = 0, \dots, 2N\}$, where

$$\bar{x}_i = \begin{cases} \dfrac{2i\sigma}{N}, & for\ \ i = 0, \dots, N/2, \\ \dfrac{i(1-2\sigma)}{N}, & for\ \ i = \dfrac{N}{2}+1, \dots, N/2 \\ \dfrac{2i\sigma}{N}, & for\ i = \dfrac{3N}{2}+1, \dots, 2N. \end{cases}$$

Let $\overline{U}(\bar{x}_i, \bar{t}_n)$solutions of the discrete problems (3.1) on the mesh $D^{2N}$using the same transition point.
Therefore, we use the following extrapolation formula
$U_{ext}(x_i,\ t_n) = \frac{1}{3}(4\overline{U} - u)(x_i, t_n)$, for $(x_i,\ t_n) \in D^N$.(4.1)

**Theorem 2**

Let u be the solution of the continuous problem (1.1) and $U_{ext}$ be the solution obtained by the Richardson extrapolation technique (4.1) by solving the discrete problem (3.1) on two meshes $D^N$and$D^{2N}$. Then we havethe following error bound associated with $U_{ext}$:
$\max_{i,n}\left|(u - U_{ext})(x_i,\ tn)\right| \le C((N^{-1}\ln N)^4 + \Delta t^4)$,
for $i = 1, \dots, N-1$.

## V. NUMERICAL RESULTS AND DISCUSSIONS

**Example 5.1**Consider the following singularly perturbed delay parabolic IBVP:

$$\begin{cases} u_t - \varepsilon u_{xx} + \frac{(1+x)^2}{2} u = t^3 - u(x,\ t-1), (x,\ t) \in (0,\ 1) \times (0,\ 2], \\ u(x,t) = 0,\ \ (x,t) \in [0,\ 1] \times [-1,\ 0], \\ u(0,t) = 0, u(1,t) = 0,\ t \in [0,\ 2]. \end{cases}$$
(5.1)

The exact solution of (5.1) is unknown. To obtain the pointwise errors and to verify the $\varepsilon$ –uniform convergence of the proposed scheme, we use the double mesh principle. Let $\widetilde{U}(x_i,\ t_n)$ be the numerical solution obtained on the fine mesh $\widetilde{D}^{2N} = \widetilde{\Omega}_x^{2N} \times \widetilde{\Omega}_t^{2M}$with $2N$mesh intervals in the spatial direction and $2M$mesh intervals in the$t$-direction, where$\Omega_x^{2N}$is piecewise-uniform Shishkin mesh as like $\Omega_x^N$with the same transition parameter. Now for each $\varepsilon$, we calculate the maximum point wise error by

$$E_\varepsilon^{N,\Delta t} = \max_{(x_i, t_n) \in D^N} \left|(U - \widetilde{U}^{2N,\Delta t})(x_i, tn)\right|$$

and the corresponding order of convergence by $P_\varepsilon^{N,\Delta t} = \log_2\left(\frac{E_\varepsilon^{N,\Delta t}}{E_\varepsilon^{2N,\Delta t/2}}\right)$.

The numerical solution of (5.1) is plotted in Figure 1 for various values of $\varepsilon$.These figures confirm the existence of boundary layers near $x = 0$and $x = 1$.The calculated maximum pointwise errors $E_\varepsilon^{N,\Delta t}$ and the rate of convergence$P_\varepsilon^{N,\Delta t}$for Example (5.1) by using central difference scheme on space and the Crank-Nicolson scheme on time scale is presented in Table 1. Clearly, it shows the dominance of the time derivative and after extrapolation, the rate of convergence is almost fourth order after extrapolation.

**Table 1:** $E_{\varepsilon}^{N,\Delta t}$ and $P_{\varepsilon}^{N,\Delta t}$ generated on S-mesh by using Crank-Nicolson scheme.

| $\varepsilon$ | Extrapolation | Number of intervals N | | | | |
|---|---|---|---|---|---|---|
| | | 32/10 | 64/40 | 128/160 | 256/640 | |
| 1e-2 | Before | 1.0812e-2<br>2.0702 | 2.5746e-3<br>2.0313 | 6.2981e-4<br>2.0212 | 1.5515e-4<br>2.0121 | 3.8465e-5 |
| | After | 1.0917e-6<br>3.8939 | 7.3444e-8<br>3.9746 | 4.6717e-9<br>3.9937 | 2.9325e-10<br>3.9984 | 1.8348e-11 |
| 1e-4 | Before | 9.0217e-3<br>1.4189 | 3.3741e-3<br>1.5281 | 1.1699e-3<br>1.5559 | 3.9791e-4<br>1.6554 | 1.2631e-4 |
| | After | 3.3446e-4<br>2.7042 | 5.1321e-5<br>3.0377 | 6.2498e-6<br>3.4893 | 5.5654e-7<br>3.9206 | 3.6751e-8 |
| 1e-6 | Before | 9.0217e-3<br>1.4189 | 3.3741e-3<br>1.5281 | 1.1699e-03<br>1.5559 | 3.9791e-4<br>1.6554 | 1.2631e-4 |
| | After | 3.3446e-4<br>2.7042 | 5.1321e-5<br>3.0377 | 6.2498e-6<br>3.2276 | 6.6720e-7<br>3.3140 | 6.7088e-8 |
| 1e-8 | Before | 9.0217e-3<br>1.4189 | 3.3741e-3<br>1.5281 | 1.1699e-3<br>1.5559 | 3.9791e-4<br>1.6554 | 1.2631e-4 |
| | After | 3.3446e-4<br>2.7042 | 5.1321e-5<br>3.0377 | 6.2498e-6<br>3.2276 | 6.6720e-7<br>3.3140 | 6.7088e-8 |

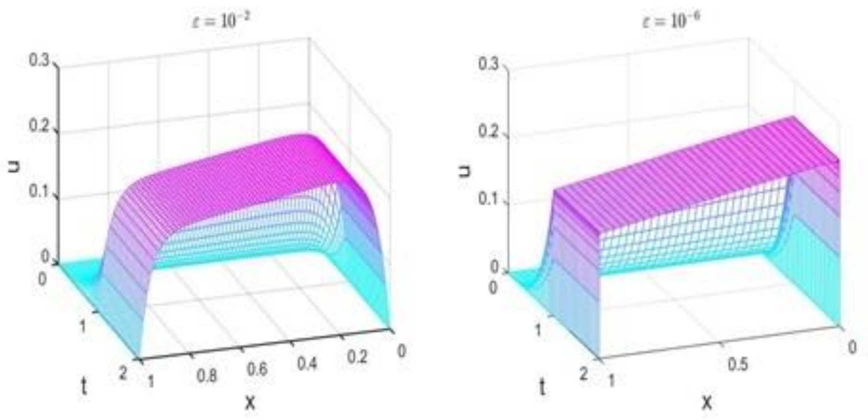


**Figure 1:** Surface plots of the numerical solution for Example 5.1.

## Conflict of Interest

The authors do not have any conflict of interest regarding the publication of the paper.

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